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J. DE VRIES

# ON THE EXISTENCE OF G-COMPACTIFICATIONS

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On the existence of G-compactifications *)

by

J. de Vries

ABSTRACT

The main result of this paper is that every Tychonoff G-space has a G-compactification provided the topological group G is locally compact. As an application we improve an embedding theorem of CARLSON.

KEY WORDS & PHRASES: G-*space, equivariant embedding, compactification.*

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*) This paper is not for review; it is meant for publication elsewhere.

## 1. INTRODUCTION

In this paper we contribute to the solution of the following problem: *Can every topological transformation group* $\langle G,X,\pi\rangle$ *with* X *a Tychonoff space equivariantly be embedded in a topological transformation group* $\langle G,Y,\sigma\rangle$ *with* Y *a compact Hausdorff space? If so, how "small" can* Y *be chosen?* Several authors have worked earlier on this problem. For the case of discrete groups (where only the second part of the problem is non-trivial) we refer to [6], [2] (section 3.4), and [1]. For the case $G = \mathbb{R}$ we mention [4], and for more general groups,[3], [11] and [13]. For a categorical motivation of this question, see [11] or [12], the final remarks in 4.3.13. Our result improves all earlier results by showing that the answer is affirmative for every locally compact group G, no matter what the Tychonoff space X and the action $\pi$ look like, and that one can choose the compact Hausdorff space Y such that $w(Y) \le \max\{w(G), w(X)\}$. Here $w(Z)$ denotes the weight of the topological space Z.

As a typical application of our result we mention the following. In [5], Theorem 1, CARLSON describes a dynamical system $(\tau, C_V^\infty)$ which is universal for the class of all dynamical systems $(\pi, X)$ on separable metrizable spaces X such that the action $\pi$ is, what he calls, *bounded* (in the sequel, we shall call his notion of boundedness: *metrical* boundedness). A consequence of the main result of our paper is that the condition of being bounded is superfluous in CARLSON's theorem: any dynamical system with a separable metrizable phase space is metrically bounded (cf. 4.1 below). No reparametrization is needed, as was suggested at the end of [5].

## 2. PRELIMINARIES

Notation will be as in [12], but for convenience we recall here some definitions and terminology. A *topological transformation group* (ttg) or a G-space is a triple $\langle G,X,\pi\rangle$ where G is a topological group, X is a topological space, and $\pi$ is a (left) *action* of G on X, that is, $\pi : G \times X \to X$ is a continuous function, $\pi(e,x) = x$ and $\pi(t,\pi(s,x)) = \pi(ts,x)$ for every $x \in X$ and $s,t \in G$ (e denotes the identity of G). The *transitions* $\pi^t : X \to X$

and the *motions* $\pi_x : G \to X$ are defined by $\pi^t x := \pi(t,x) =: \pi_x t$ for $(t,x) \in G \times X$. If $\langle G,X,\pi\rangle$ and $\langle G,Y,\sigma\rangle$ are G-spaces, then a function $f : X \to Y$ is called *equivariant* whenever $f\circ\pi^t = \sigma^t\circ f$ for every $t \in G$. If f is an equivariant dense topological embedding of X into Y, and Y is a compact Hausdorff space, then we call $\langle G,Y,\sigma\rangle$ a G-*compactification of* $\langle G,X,\pi\rangle$. A necessary condition for $\langle G,X,\pi\rangle$ to have a G-compactification is that X is a Tychonoff space and from now on we assume that X is such a space. Recall that X admits a uniformity which is compatible with the topology of X (shortly: a uniformity *for* X). We shall call a ttg $\langle G,X,\pi\rangle$ $\mathcal{U}$*-bounded* or *bounded w.r.t.* $\mathcal{U}$ whenever $\mathcal{U}$ is a uniformity for X and

$$\forall\alpha \in \mathcal{U},\ \exists U \in \mathcal{V}_e \vdots (\pi^t x,x) \in \alpha \quad \text{for all } t \in U,\ x \in X$$

(here $\mathcal{V}_e$ denotes the neighbourhood filter of e in G). In this case we shall also call the action $\pi$ of G on X bounded w.r.t. $\mathcal{U}$ or $\mathcal{U}$-bounded. The relevance of the notion of boundedness for the problem of the existence of G-compactifications is immediate from the following result, which generalizes Theorem 3.1(1) of [3]:

2.1. THEOREM. *Let* $\langle G,X,\pi\rangle$ *be a ttg with* X *a Tychonoff space. The following conditions are equivalent:*

(i) *There exists a* G-*compactification* $\langle G,Y,\sigma\rangle$ *of* $\langle G,X,\pi\rangle$.

(ii) *The action* $\pi$ *of* G *on* X *is bounded w.r.t. some uniformity* $\mathcal{U}$ *for* X.

*If these conditions are fulfilled, then* $\langle G,X,\pi\rangle$ *has a* G-*compactification* $\langle G,Y_0,\sigma_0\rangle$ *such that*

$$(*) \qquad w(Y_0) \le \max\{w(G),\ w(X)\}.$$

PROOF. Cf. [12], Proposition 7.3.12. □

2.2. REMARK. The proof of (i) ⇒ (ii) is easy. A quick proof of the converse imlication (ii) ⇒ (i), using Theorem 3.1 of [3], is included in [13]; the difference between the result in [3] and ours is, that in [3] condition (ii) is replaced by the stronger condition

(ii)' The action $\pi$ is bounded w.r.t. some uniformity $\mathcal{U}$ for X and each $\pi^t$ ($t\in G$) is a unimorphism w.r.t. $\mathcal{U}$.

In addition, [3] and [13] contain no proof of the inequality (*).

## 3. MAIN RESULT

Throughout this section $\langle G,X,\pi\rangle$ shall denote a ttg with G an arbitrary locally compact Hausdorff group and X a Tychonoff space. The basic result is:

3.1. PROPOSITION. *There exists a uniformity $\mathcal{U}$ for X such that the action $\pi$ of G on X is $\mathcal{U}$-bounded.*

PROOF. It is sufficient to prove that there exists a set $\{g_i : i \in I\}$ of continuous, [0,1]-valued functions on X satisfying the conditions

(1) $\{g_i : i \in I\}$ separates points and closed subsets of X, i.e. for any closed subset A of X and $x \in X \sim A$ there exists $i \in I$ with $g_i(x) \notin \mathrm{cl}\, g_i[A]$.

(2) $\forall i \in I : \{g_i \circ \pi_x : x \in X\}$ is equicontinuous at e.

Indeed, if we have such a family, let $\mathcal{U}$ denote the weakest uniformity on X making every $g_i : X \to [0,1]$ uniformly continuous. Then the topology generated by $\mathcal{U}$ coincides with the weakest topology on X, making every $g_i$ continuous. By (1), this topology is just the original topology of X, hence $\mathcal{U}$ is a uniformity for X. In addition, $\pi$ is $\mathcal{U}$-bounded. For if $\alpha \in \mathcal{U}$, then there are a finite subset $I_\alpha$ of I and a real number $\varepsilon > 0$ such that for all $(x,y) \in X \times X$:

$$|g_i(x) - g_i(y)| < \varepsilon \quad \text{for all } i \in I_\alpha \quad \Rightarrow \quad (x,y) \in \alpha.$$

In view of (2), there is for every $i \in I$ a neighbourhood $U_i$ of e in G such that $|g_i(\pi(t,x)) - g_i(x)| < \varepsilon$ for every $t \in U_i$ and $x \in X$. Hence $(\pi(t,x),x) \in \alpha$ for all $t \in \bigcap\{U_i : i \in I_\alpha\}$ and $x \in X$, as desired.

We shall demonstrate now that a family $\{g_i : i \in I\}$ with properties (1) and (2) exists. The proof will be interrupted by several lemmas.

<u>3.2. LEMMA</u>. *There exists a set* $\Phi$ *of left uniformly continuous, real valued functions on* G *such that*

(3) $\forall\varphi \in \Phi : \varphi(e) = 0 \ \& \ \varphi(t) \geq 0 \ (t \in G)$ ;

(4) $\forall t \in G \, [t \neq e \Rightarrow \exists \varphi \in \Phi \,\vdots\, \varphi(t) > 0)]$ ;

(5) $\forall\varphi \in \Phi$ : the set $A_\varphi := \{t \in G \,\vdots\, \varphi(t) \leq 2\}$ *is a compact subset of* G.

<u>PROOF</u>. Since G is a locally compact Hausdorff group, there exists a local base $\mathcal{B}$ at e such that cl U is compact for every $U \in \mathcal{B}$. Choose for every $U \in \mathcal{B}$ a continuous function $\varphi_U : G \to [0,3]$ such that $\varphi_U(e) = 0$ and $\varphi_U(t) = 3$ if $t \in G \sim U$. Now take $\Phi := \{\varphi_U \,\vdots\, U \in \mathcal{B}\}$. Since for each $U \in \mathcal{B}$, $\varphi_U$ is constant outside a compact set (viz.cl U) it is clear that $\varphi_U$ is left uniformly continuous. □

3.3. If $\Phi$ is as above and if for $(n,\varphi) \in \mathbb{N} \times \Phi$ we define

$$U_{n,\varphi} := \{t \in G \,\vdots\, \varphi(t) \leq 1/n\},$$

then $U_{n,\varphi}$ is a compact neighbourhood of e in G, and

$$\cap\{U_{n,\varphi} \,\vdots\, (n,\varphi) \in \mathbb{N} \times \Phi\} = \{e\}.$$

*It follows easily that* $\{U_{n,\varphi} \,\vdots\, (n,\varphi) \in \mathbb{N} \times \Phi\}$ *is a local base at* e (see e.g.[8], the proof of 8.5, which can easily be adepted to the present situation). In particular, if $\Phi$ is countable, then G is metrizable (cf.[8],8.3). Conversely, if G is metrizable, one can choose $\Phi$ such that it contains only one element: set $\varphi(t) := d(e,t)$ $(t \in G)$ where d is a left invariant metric for G such that $\{t \in G \,\vdots\, d(e,t) \leq 2\}$ is compact.

3.4. Fix a set $\Phi$ as indicated in 3.2. For every $f \in C(X,[0,1])$ and $\varphi \in \Phi$, a real-valued function $\tilde{f}_\varphi$ on X can be defined by

(6) $$\tilde{f}_\varphi(x) := \inf_{t \in G} \{\varphi(t) + f(\pi^t x)\}$$

for $x \in X$. Incidentally, this definition and the lemmas 3.5 and 3.7 below are motivated by Lemma 7 in [7].

3.5. LEMMA. *The functions* $\tilde{f}_\varphi$ ($f \in C(X,[0,1])$ *and* $\varphi \in \Phi$) *map* X *continuously into the interval* $[0,1]$.

PROOF. Clearly, $0 \leq \tilde{f}_\varphi(x) \leq \varphi(e) + f(x) = f(x) \leq 1$ for every $x \in X$. So we need only to prove continuity of $\tilde{f}_\varphi$. In order to do so, first observe that for every $t \in G$ with $\varphi(t) \geq 2$ we have $\varphi(t) + f(\pi^t x) \geq 2 > 1 \geq \tilde{f}_\varphi(x)$. Consequently, with $A_\varphi$ as defined in (5), we have

$$(7) \qquad \tilde{f}_\varphi(x) = \inf_{t \in A_\varphi} \{\varphi(t) + f(\pi^t x)\}.$$

However, the function $t \mapsto \varphi(t) + f(\pi^t x) : A_\varphi \to \mathbb{R}$ is continuous and $A_\varphi$ is compact. Hence the infinum in (7) is not only actually attained at some point $t_x \in A_\varphi$ but it follows also that $\tilde{f}_\varphi$ is continuous, as is well-known and easy to prove. □

3.6. LEMMA. *If* $f(x) = 0$ *then* $\tilde{f}_\varphi(x) = 0$ *for every* $\varphi \in \Phi$. *If* $f(x) > 0$ *then there exists* $\varphi \in \Phi$ *such that* $\tilde{f}_\varphi(x) > 0$.

PROOF. If $f(x) = 0$, then the inequalities $0 \leq \tilde{f}_\varphi(x) \leq f(x)$ (cf. the proof of 3.5) imply that $\tilde{f}_\varphi(x) = 0$. If $f(x) > 0$, then there is $U \in \mathcal{V}_e$ such that $f(\pi^t x) > \frac{1}{2}f(x)$ for all $t \in U$. By 3.3, there exists $\varphi \in \Phi$ and $n \in \mathbb{N}$ such that $U_{n,\varphi} \subseteq U$. We may and shall assume that $1/n \leq \frac{1}{2}f(x)$. Then we have for every $t \in G$, $\varphi(t) + f(\pi^t x) > 1/n$, whence $\tilde{f}_\varphi(x) \geq 1/n > 0$. □

3.7. LEMMA. *For every* $f \in C(X,[0,1])$ *and* $\varphi \in \Phi$, *the family* $\{\tilde{f}_\varphi \circ \pi_x : x \in X\}$ *is equicontinuous at* e.

PROOF. Fix f and $\varphi$ as indicated. For every $(t,x) \in G \times X$ we have

$$\begin{aligned}\tilde{f}_\varphi(\pi(t,x)) &= \inf_{s\in G} \{\varphi(s) + f(\pi(st,x))\} \\ &= \inf_{u\in G} \{\varphi(ut^{-1}) - \varphi(u) + \varphi(u) + f(\pi^u x)\} \\ &\geq \inf_{u\in G} \{\varphi(ut^{-1}) - \varphi(u)\} + \tilde{f}_\varphi(x)\end{aligned}$$

Since $\varphi$ is left uniformly continuous on G, there is for every $\varepsilon > 0$ a neighbourhood $U_\varepsilon$ of e in G such that $|\varphi(ut^{-1}) - \varphi(u)| < \varepsilon$ for all $t \in U_\varepsilon$ and $u \in G$.

Consequently, $\tilde{f}_\varphi(\pi(t,x)) \geq \tilde{f}_\varphi(x) - \varepsilon$ for all $t \in U_\varepsilon$ and all $x \in X$. Similarly, there is $V_\varepsilon \in \mathcal{V}_e$ such that $\tilde{f}_\varphi(x) \geq \tilde{f}_\varphi(\pi(t,x)) - \varepsilon$ for all $t \in V_\varepsilon$ and all $x \in X$. Hence

$$(8) \qquad |\tilde{f}_\varphi(\pi(t,x)) - \tilde{f}_\varphi(x)| < \varepsilon$$

for every $t \in U_\varepsilon \cap V_\varepsilon$ and every $x \in X$. □

3.8. In the preceding proof we have shown a little bit more than was actually needed; namely, if $t \in U_\varepsilon \cap V_\varepsilon$ then (8) holds not only uniformly in $x \in X$, but also uniformly in $f \in C(X,[0,1])$. Hence $\{\tilde{f}_\varphi \circ \pi_x : x \in X \text{ \& } f \in C(X,[0,1])\}$ is equicontinuous at e. However, the statement of lemma 3.7 is sufficient for our purposes.

3.9. PROOF OF 3.1 (continued). Consider the family $\{\tilde{f}_\varphi : (\varphi,f) \in \Phi \times C(X,[0,1])\}$. By 3.5, this is a set of continuous, $[0,1]$-valued functions, and it is easy to see that it satisfies condition (1) (use lemma 3.6 and the fact that for any closed set $A \subseteq X$ and any point $x \in X \sim A$ there is $f \in C(X,[0,1])$ with $f(x) = 1$ and $f[A] = \{0\}$). In addition, our family fulfills condition (2): this is exactly lemma 3.7. □

3.10 THEOREM. *Any ttg* $\langle G,X,\pi\rangle$ *with* G *a locally compact Hausdorff group and* X *a Tychonoff space has a* G-*compactification* $\langle G,Y,\sigma\rangle$. *Moreover, one may assume that*

$$w(Y) \leq \max\{w(G), w(X)\}.$$

PROOF. Combine 2.1 and 3.1. □

3.11. The restriction that G is Hausdorff can be omitted from 3.1 and 3.10 This can be seen as follows. Suppose we are given a ttg $\langle H,X,\pi'\rangle$ with H locally compact but not Hausdorff, and X a Tychonoff space. Then the stability subgroup $H_0 := \{t \in H : \pi'(t,x) = x \text{ for every } x \in X\}$ is a closed normal subgroup of H, hence $G := H/H_0$ is a locally compact *Hausdorff* topological group. Let $\pi$ denote the naturally induced action of G on X. Then theorem 3.10 can be applied to $\langle G,X,\pi\rangle$ so as to produce a G-compactification

$\langle G,Y,\sigma\rangle$ of $\langle G,X,\pi\rangle$. If $\psi\colon H \to G$ is the quotient mapping, then an action $\sigma^{\psi}$ of H on Y can be defined by $\sigma^{\psi}(t,y) := \sigma(\psi(t),y)$ for $(t,y) \in H \times Y$. It is plain that now $\langle H,Y,\sigma^{\psi}\rangle$ is the desired H-compactification of $\langle H,X,\pi'\rangle$..

## 4. AN APPLICATION

In [5], a dynamical system (that is, an $\mathbb{R}$-space in our terminology) is described, which is defined by a Cauchy problem for an autonomous partial differential equation and which has the following property: every "bounded" dynamical system on a separable metrizable space can equivariantly be embedded in this "universal" system. However, the notion of boundedness which occurs in [5] differs slightly from ours, and we shall call it therefore *metrical* boundedness. Here is the definition: a ttg $\langle G,X,\pi\rangle$ with X a metrizable space is called *metrically bounded w.r.t. a metric* d provided it is bounded w.r.t. the uniformity $\mathcal{U}_d$ which corresponds with d. Here the situation is somewhat subtle: a bounded action on a metrizable space X (w.r.t. some uniformity $\mathcal{U}$ for X) may be *not* metrically bounded w.r.t. any metric d for X, even if the acting group is a separable locally compact group (cf.[10], p.110, where an example is given with a $\sigma$-compact locally compact Hausdorff group G; if we take in that example for the index set A a set with the cardinality of the continuum, we obtain a separable group: a product of continuously many separable spaces is still separable). However, if G is a $\sigma$-compact locally compact Hausdorff group (in particular, if $G = \mathbb{R}$) and X is a *separable* metrizable space, then boundedness of $\langle G,X,\pi\rangle$ w.r.t. some uniformity $\mathcal{U}$ implies metric boundedness of $\langle G,X,\pi\rangle$ w.r.t. some metric d.

For a proof of this fact in its full generality, we refer to [10], Corollary 4.11, or to [12], 7.3.14. For the special case of $G = \mathbb{R}$ we present here a quick proof:

<u>4.1. PROPOSITION</u>. *Every ttg* $\langle \mathbb{R},X,\pi\rangle$ *with* X *a separable metrizable space is metrically bounded w.r.t. some metric* d *for* X.

<u>PROOF</u>. According to 3.10, the ttg $\langle \mathbb{R},X,\pi\rangle$ has an $\mathbb{R}$-compactification $\langle \mathbb{R},Y,\sigma\rangle$ with $w(Y) \le \max\{w(\mathbb{R}), w(X)\} = \aleph_0$. Hence Y is metrizable. Clearly, the action of $\sigma$ of $\mathbb{R}$ on Y is bounded w.r.t. any metric d for Y, hence the

action $\pi$ of $\mathbb{R}$ on X is bounded w.r.t. the restriction of d to X. □

4.2. COROLLARY. *Every ttg* $\langle \mathbb{R}, X, \pi\rangle$ *with* X *a separable metrizable space can equivariantly be embedded in* CARLSON'*s universal system* $\langle \mathbb{R}, C_V^\infty, \tau\rangle$.

PROOF. Use proposition 4.1 above and [5], Theorem 1. □

4.3. REMARK. In [5], boundedness is used only in order to prove that the equivariant embedding mapping F constructed there is actually a relatively open mapping: for injectivity and continuity of F no boundedness condition is needed. Hence a different proof of 4.2 can be given as follows: if $\langle \mathbb{R}, X, \pi\rangle$ is a ttg with X a separable metrizable space, then there is an $\mathbb{R}$-compactification $\langle \mathbb{R}, Y, \sigma\rangle$ with Y compact metrizable and also separable (cf. the proof of 4.1). Apply CARLSON's proof to $\langle \mathbb{R}, Y, \sigma\rangle$; note that a continuous injection F of Y into $C_V^\infty$ is automatically a topological embedding (Y is compact and $C_V^\infty$ is Hausdorff). Hence the restriction of F to X is a topological embedding of X in $C_V^\infty$.

A similar application of theorem 3.10 to another embedding problem is the following one: Let G be an infinite $\sigma$-compact, locally compact Hausdorff group. In [9] we constructed a linear action $\pi$ of G on the Hilbert space $L^2(G\times G)$ such that every bounded ttg $\langle G, X, \sigma\rangle$ with X a separable metrizable space can equivariantly be embedded in $\langle G, L^2(G\times G), \pi\rangle$. By 3.10, we can remove here the boundedness condition as well, provided G is second countable, i.e. separable and metrizable. For such groups G we infer that every separable metrizable G-space can equivariantly be embedded in the Hilbert G-space $\langle G, L^2(G\times G), \pi\rangle$.

Further applications of 3.10 will be published in the future.

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